The University of Georgia
Department of Mathematics and Science Education
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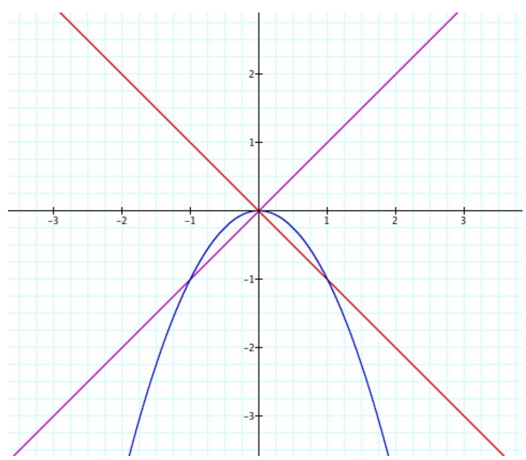
EMAT 6680 - Assignment 1

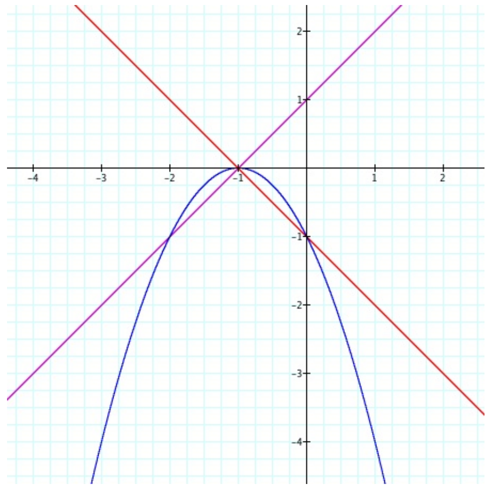
Question: Find two linear functions $f(x)$ and $g(x)$ such that their product $h(x) = f(x)g(x)$ is tangent to each of $f(x)$ and $g(x)$ at two distinct points.

By Brandon Samples

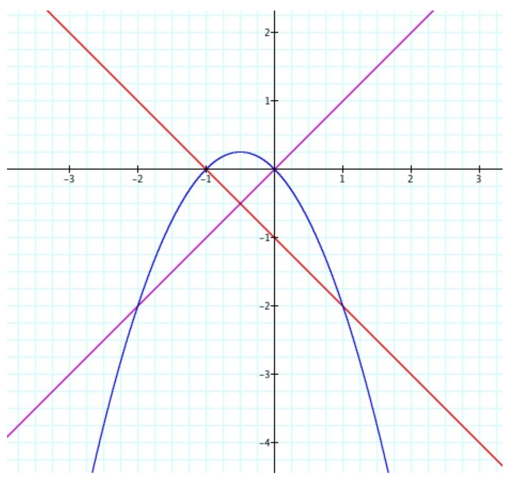
To begin this assignment, let's explore a few graphs. If we let $f(x) = x$ and $g(x) = -x$, then the graphs of f, g, and h are given in the following figure:



Upon inspection, it seems to be the case that the lines will not be tangent to the graph of $h(x)$ at two distinct points as long as the two lines pass through the same point of the x-axis. As another example of this, consider $f(x) = x + 1$ and $g(x) = -x - 1$ with graph:



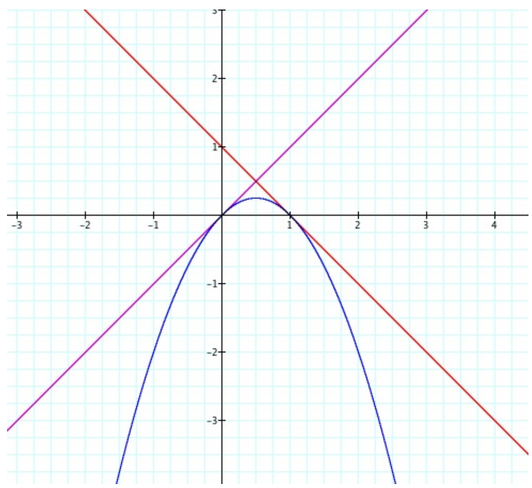
This leads us to considering two linear functions $f(x)$ and $g(x)$ where f and g do not have the same point of intersection with the x-axis. Let's try the functions $f(x) = x$ and $g(x) = -x - 1$ with graph:



However, we are still not getting lines which are tangent to the graph of h, but rather are secant lines through two points. Let's try to think about this another way. Notice that the above graph indicates the following behavior (which will always be true): If the two lines have nonzero slope (one positive and one negative) and intersect away from the x-axis, then the lines will pass through the x-axis and these distinct points will also intersect the graph of $h(x)$. Now, if these points are to be the points of tangency of $f(x)$ and $g(x)$ to $h(x)$, then it follows that the only points of intersection must occur at the x-axis, i.e., the points of tangency must occur on the x-axis. Let's explore this idea.

Suppose that we want the two points of tangency to occur at $P = (0, 0)$ and $Q = (1, 0)$. Let $f(x) = ax + b$ and $g(x) = cx + d$ and note that the above discussion tells us that $f(0) = 0$ and $g(1) = 0$. As a result, we see that $b = 0$ and $c + d = 0$, hence $f(x) = ax$ and $g(x) = cx - c = c(x - 1)$. Now, the function $h(x) = f(x)g(x) = ac(x^2 - x)$, so the derivative is given by $h'(x) = ac(2x - 1)$. By construction, $h'(0) = a$

and $h'(1) = c$ since the lines f and g are supposed to be tangent to h at P and Q. Therefore, $-ac = a$ and $ac = c$, which implies that $0 = a(c+1)$ and $0 = c(a-1)$. If we choose $a = 1, b = 0, c = -1$, and $d = 1$, then $f(x) = x$ and $g(x) = -x + 1$. It is easy to check that $f(x)$ and $g(x)$ have the required properties and they are indicated in the following graph:



Finally, we might suspect that this process works for any two distinct points of the form $P = (x_1, 0)$ and $Q = (x_2, 0)$.

Lemma 1. *For any two points $P = (x_1, 0)$ and $Q = (x_2, 0)$, there exists distinct linear functions $f(x)$ and $g(x)$ such that $f(x)$ and $g(x)$ are tangent to the graph of the product function $h(x) = f(x)g(x)$ at the points P and Q.*

Proof. Let $f(x) = ax + b$ and $g(x) = cx + d$ and note that the hypothesis forces the following equations:

$$0 = f(x_1) = ax_1 + b$$
$$0 = g(x_1) = cx_1 + d.$$

Consequently, $b = -ax_1$ and $d = -cx_2$, so $f(x) = ax - ax_1$ and $g(x) = cx - cx_2$. Next, the product function h is given by $h(x) = acx^2 + (-acx_2 - acx_1)x + acx_1x_2$, which has a derivative given by $h'(x) = ac(2x - x_1 - x_2)$. Since P and Q are the points of tangency, we obtain the equations:

$$a = h'(x_1) = ac(x_1 - x_2)$$
$$c = h'(x_2) = ac(x_2 - x_1),$$

which implies that

$$0 = a(cx_1 - cx_2 - 1)$$
$$0 = c(ax_2 - ax_1 - 1).$$

Therefore, we can let a, b, c and d be given by

$$a = \frac{1}{x_2 - x_2}$$
$$b = \frac{-x_1}{x_2 - x_1}$$
$$c = \frac{1}{x_1 - x_2}$$
$$d = \frac{-x_2}{x_1 - x_2},$$

and notice that

$$f(x) = \frac{x}{x_2 - x_2} - \frac{x_1}{x_2 - x_1}$$

and

$$g(x) = \frac{x}{x_1 - x_2} - \frac{x_2}{x_1 - x_2}$$

represent the desired linear functions. □